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Diffraction of ocean waves around a hollow cylindrical shell structure

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ABSTRACT

In recent years, there has been renewed interest in problems of diffraction and radiation of ocean waves around structures, in relation to "green" power generation by Oscillating Water Column (OWC) devices. In this paper, we present a first-order analytical solution for the diffraction of ocean waves around a hollow cylindrical shell structure suspended in an ocean of finite depth. By revisiting work done by Garrett (1970) on the problem of a bottomless harbor, but adopting a different and more direct method, we obtain the solution for the diffracted wave potential.

Using the new approach, we analyze the dependence of the solution upon various parameters, as well as the rate of convergence of the series solution. Apart from some problems we observed with matching the boundary condition at the edge of the cylinder, we find good agreement with Garrett's results. Furthermore, we analyze the accuracy of the solution as a function of cylinder submergence. Finally, we briefly discuss the extension of the method to the related problem of radiation of surface waves by an oscillating surface pressure inside a hollow suspended cylindrical shell structure.

The results presented in this paper show that even a simple hollow cylinder, which only captures the most essential feature of an OWC, can produce very complicated patterns of diffracted waves. This clearly demonstrates the complexity associated with using OWC devices to convert ocean wave energy to electricity and the necessity of further fundamental research needed in this area before we can realistically adopt this new technology to *efficiently* produce renewable energy in practice.

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1. Introduction

Utilizing renewable energy resources has become an important field of study with regards to combatting climate change. Recent studies show that Oscillating Water Column (OWC) devices are an attractive approach to convert the power of ocean surface waves to electrical energy or to directly use the converted mechanical energy to desalinate water [1]. Such devices work on the principle that a plane wave field that is incident upon a hollow vertical chamber will force air to oscillate inside the chamber, which can be used to drive a specially designed turbine, that rotates in one direction while the air flow that drives its rotation alternates its direction. For example, Oceanlinx Limited Australia has already developed a prototype based on the concept of OWC. Fig. 1 displays this prototype, which is currently installed at Port Kembla, NSW, Australia and has successfully converted ocean wave energy into electricity in a number of test runs.

Two of the problems that arise when trying to design an OWC plant to efficiently extract the maximum energy from an incoming wave field are:

1. How to design the shape of the OWC chamber to extract the largest possible oscillating pressure inside the device.

2. Where to position the device in the ocean to capture waves as large as possible.

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Fig. 1. An OWC prototype located at Port Kembla, NSW, Australia.

The first point relates to diffraction of incoming plane waves inside the cylinder – we would like to know the optimal design for which the wave elevation inside the cylinder is a maximum, that is for which a wave would resonate and interfere constructively inside the cylinder. The second relates to the combined wave radiation and diffraction by an oscillating surface pressure inside the cylinder – we would like to plot the superimposed radiated and diffracted wave field along with incident plane wave field to fully understand how the OWC structure affects its environment.

Various mathematical treatments of problems relating to OWC devices have appeared in the literature over the years. A fundamental contribution was made by Garrett [2], who solved the first-order diffraction problem for a hollow suspended cylinder in an ocean of finite depth. Some authors have also analyzed simple two-dimensional models for wave extraction by OWC devices. There are examples in both the linear [3] and non-linear [4] regimes, and some extensions have been made to three-dimensional models [5,6]. Various numerical analyses have been performed [7–9], however, these solutions all have the disadvantage of the introduction of discretization errors due to the numerical scheme being implemented. We present an exact analytical solution, meaning the only errors introduced are from rounding.

In this paper, we present a first-order analytical solution for the problem of surface wave diffraction around a hollow suspended cylinder in an ocean of finite depth. We revisit the work done by Garrett, by using a different approach to the variational approximation and least-squares minimization technique he used to solve the problem. We find that it is possible to formulate the problem as a matrix equation, which we solve to obtain the solution directly. This approach has the advantage of requiring less work in analysis than that required by Garrett, and is also computationally easier and more elegant to implement. The results presented in this paper show that even a simple hollow cylinder, which only captures the most essential feature of an OWC, can produce very complicated patterns of diffracted waves. This clearly demonstrates the complexity associated with using OWC devices to convert ocean wave energy to electricity and the necessity of further fundamental research needed in this area before we can realistically adopt this new technology to *efficiently* produce renewable energy in practice.

The paper is divided into five sections. In Section 2 we formulate the problem in terms of an infinite set of linear equations, which we solve by means of our matrix method in Section 3. In Section 4 we present results for the diffraction problem, focussing on an analysis of: (a) convergence of the solution, (b) comparison with Garrett's results and dependence on various parameters, and (c) matching the boundary conditions. Finally, in Section 5 we present our conclusions, and discuss the extension of the problem to a combined diffraction-radiation problem, which will be the subject of a forthcoming paper.

2. Formulation

Consider a plane wave of amplitude ζ_0 and frequency σ that is incident upon a hollow cylinder of radius *a* which is suspended at height *h* above the floor of an ocean of depth *d*. We fix a coordinate system with the origin coincident with the centre of the cylinder on the ocean floor, with the *x*-direction pointing in the direction of the incoming wave, and the *z*-direction pointing vertically upwards. A schematic diagram for the diffraction problem is shown in Fig. 2. We solve

$$abla^2\phi=0,$$



Fig. 2. Schematic diagram.

subject to

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad \leqslant z = 0, \tag{2}$$

$$\frac{\partial \phi}{\partial z} - \frac{\sigma^2}{g} \phi = 0 \quad \text{on} \quad z = d, \tag{3}$$

$$\frac{\partial \phi}{\partial z} = z \quad \text{on} \quad z = d, \tag{3}$$

$$\frac{\partial \varphi}{\partial r} = 0 \quad \text{on } r = a, \quad \text{for } h \leqslant z \leqslant d,$$
(4)

where ϕ is the velocity potential and g is the gravitational acceleration. Eqs. (2) and (4) are zero displacement boundary conditions for the solid ocean floor and cylinder surface, respectively, while Eq. (3) is the combined kinematic and dynamic free surface condition at the ocean surface. Using the cylindrical symmetry of the problem, we can write ϕ as

$$\phi(r,z,\theta) = \zeta_0 \sum_{m=0}^{\infty} \epsilon_m i^m \psi_m(r,z) \cos m\theta, \tag{5}$$

and define the surface disturbance $\boldsymbol{\zeta}$ as

$$\zeta(r,\theta) = \zeta_0 \sum_{m=0}^{\infty} \epsilon_m i^m \chi_m(r) \cos m\theta, \tag{6}$$

where ϵ_m is equal 1 when m = 0, and equal to 2 otherwise. The functions χ_m and ψ_m are related by the kinematic surface condition

$$\chi_m(r) = \frac{\partial \psi_m}{\partial z} \Big|_{z=d}.$$
(7)

As the problem naturally consists of two domains, an inner domain within the boundary of the shell wall and outer domain that extends from the shell wall to infinity, we decompose ϕ into two functions defined on $r \leq a$ and $r \geq a$, respectively:

$$\phi = \begin{cases} \phi_{\text{int}}, & r \leq a, \\ \phi_{\text{ext}}, & r \geq a, \end{cases}$$
(8)

where ϕ_{int} and ϕ_{ext} are referred to as the interior and exterior wave potential, respectively, hereafter. Consequently, we need to introduce two continuity conditions:

$$\phi_{\text{int}} = \phi_{\text{ext}}, \quad \text{and}$$

$$(9)$$

$$\frac{\partial \phi_{\text{int}}}{\partial r} = \frac{\partial \phi_{\text{ext}}}{\partial r} \quad \text{at } r = a \quad \text{for } 0 \leqslant z \leqslant h, \tag{10}$$

which are necessary to close the system.

The solution of the governing equation (1) in cylindrical coordinates can be written as an expansion in terms of cylindrical Bessel functions. We expand ψ_m in the interior region as

$$\psi_{m,\text{int}}(r,z) = [C_1 J_m(kr) + C_2 Y_m(kr)] Z_k(z) + \sum_{\alpha} [C_{1\alpha} I_m(\alpha r) + C_{2\alpha} K_m(\alpha r)] Z_{\alpha}(z),$$
(11)

where C_1 , C_2 , $C_{1\alpha}$ and $C_{2\alpha}$ are constants to be determined through the satisfaction of the boundary conditions. The exterior function $\psi_{m,\text{ext}}$ has exactly the same form of expansion, except the coefficients in front of each of the eigenfunctions are different. The wave numbers k and α are determined from the dispersion relations

$$\omega^2 - gk \tanh kd = 0, \text{ and}$$
(12)
$$\omega^2 + \alpha g \tan \alpha d = 0,$$
(13)

respectively. Dispersion relation (12) has a single pair of real roots $\pm k$ corresponding to the propagating wave, while dispersion relation (13) has an infinite set of real roots $\pm \alpha_n$, corresponding to the non-propagating evanescent modes of the solution. It is the existence of this infinite set of evanescent modes which provides the theoretical basis for the series solution we will employ below, and accounts for the summation over α in (11). Numerically, the terms generated by the wave numbers from dispersion relation (13) are small and are associated with localized waves near the cylinder only, however, they must be included for completeness of the solution. Mathematically, the dispersion relations (12) and (13) are a result of satisfying the free-surface boundary condition (3), while physically they present a most unique feature of water waves, i.e., wave velocity is dependent on wave number and thus waves of different wave numbers will "disperse" or spread out over time.

The functions Z_k and Z_{α} in (11) are the associated eigenfunctions in the vertical direction, corresponding to each mode k and α , respectively. They are of the form

$$Z_k(z) = N_k^{-\frac{1}{2}} \cosh kz, \tag{14}$$

$$Z_{\alpha}(z) = N_{\alpha}^{2} \cos \alpha z, \tag{15}$$

where

$$N_{k} = \frac{1}{2} \left(1 + \frac{\sinh 2kd}{2kd} \right),$$
$$N_{\alpha} = \frac{1}{2} \left(1 + \frac{\sin 2\alpha d}{2\alpha d} \right).$$

Since $\{Z_k, Z_{\alpha}\}$ form an orthonormal set in [0, d] [10], we may expand the function $\frac{\partial \psi_m}{\partial r}$ over the entire interval [0, d] as

$$\left.\frac{\partial\psi_m}{\partial r}\right|_{r=a} = \sum_{\alpha} \mathscr{F}_{m\alpha} Z_{\alpha}(z),\tag{16}$$

where

$$\mathscr{F}_{m\alpha} = \frac{1}{d} \int_0^{\pi} f_m(z) Z_{\alpha}(z) \, \mathrm{d}z, \tag{17}$$

$$f_m(z) = \frac{\partial \psi_m}{\partial r}, \quad \text{at } r = a \quad \text{for } 0 \le z < h.$$
 (18)

We thus have expansions for the interior and exterior functions

$$\psi_{m,\text{int}}(r,z) = \sum_{\alpha} \mathscr{F}_{m\alpha} \frac{I_m(\alpha r)}{\alpha I'_m(\alpha a)} Z_{\alpha}(z)$$
(19)

and

$$\psi_{m,ext}(r,z) = \left\{ J_m(kr) - \frac{J'_m(ka)}{H'_m(ka)} H_m(kr) \right\} \frac{Z_k(z)}{Z'_k(d)} + \sum_{\alpha} \mathscr{F}_{m\alpha} \frac{K_m(\alpha r)}{\alpha K'_m(\alpha a)} Z_\alpha(z),$$
(20)

respectively.

3. Solution

The constants $\mathscr{F}_{m\alpha}$ are obtained by imposing the continuity condition (9), utilising the orthogonality of the eigenfunctions Z_{α} and Z_{β} . This yields a set of infinitely many linear equations for $\mathscr{F}_{m\alpha}$ as

$$F_m C_\beta = \sum_{\alpha} E_{\beta \alpha} \mathscr{F}_{m\alpha}, \tag{21}$$

where

$$C_{\beta} = \frac{1}{d} \int_0^h Z_k(z) Z_{\beta}(z) \, \mathrm{d}z,$$
(22)

$$D_{\beta\alpha} = \frac{1}{d} \int_{0}^{h} Z_{\alpha}(z) Z_{\beta}(z) \,\mathrm{d}z,\tag{23}$$

$$E_{\beta\alpha} = (R_{\alpha} - 1)D_{\beta\alpha} + \delta_{\beta\alpha}, \tag{24}$$

$$F_m = 2\mathbf{i} \left[\pi k a^2 H'_m(ka) Z'_k(d) \right] \quad , \tag{25}$$

$$R_\alpha = \left[\alpha^2 a^2 I'_m(\alpha a) K'_m(\alpha a) \right]^{-1} . \tag{26}$$

The integrals in (22) and (23) can be evaluated exactly, and the solution of (21) for $\mathscr{F}_{m\alpha}$ gives the complete solution to the problem.

From here on we use a simpler and more direct method to solve (21) for the unknown constants $\mathscr{F}_{m\alpha}$ than that employed by Garrett. Garrett's approach is to further manipulate (21) in order to separate it into real and imaginary parts. He then uses a least-squares minimization technique on a new function Φ_{α} derived from (21) to obtain a linear equation which my be solved numerically for any number of eigenvalues α . As the exact solution is presented as an infinite summation over the roots α , Garrett truncates the summation after *N* terms to obtain an approximate solution, and investigates the dependence upon *N*. He finds Φ_k "to be almost linearly dependent on 1/*N* for large *N*", and so uses a linear extrapolation from the results for N = 20 and N = 40 to approximate the exact solution at 1/N = 0. Garrett finds that his results agree well with a smooth extrapolation using all values of *N*, except "in a situation where the answer for N = 40 was innaccurate by 15%". With this in mind, we will treat Garrett's numerical solution as a good approximation to the true solution, but should not be surprised or overly concerned if our results do not agree exactly.

Our method relies on recognizing that (21) may be formulated as a matrix equation for each value of m, and that by truncating the summations over m after M terms and α after N terms, where M and N are suitably large, we obtain an approximation that is similar to Garrett's solution. The matrix equation we solve for each m is

$$\mathbf{E}_{\boldsymbol{\beta}\boldsymbol{\alpha}}\mathscr{F}_{\boldsymbol{m}\boldsymbol{\alpha}} = F_{\boldsymbol{m}}\mathbf{C}_{\boldsymbol{\beta}},\tag{27}$$

where $\mathbf{E}_{\beta\alpha}$ is an $N \times N$ matrix, $\mathscr{F}_{m\alpha}$ is a vector with N elements, F_m is a scalar which depends upon m, and C_β as another vector with N elements. The numerical values of the entries are given by (22)–(26).

4. Results

To determine how accurate and useful our solution is, we wish to analyze:

- 1. Rate of convergence of the series solution.
- 2. Agreement with Garrett's results.
- 3. How well the boundary conditions are satisfied.

Unless otherwise indicated, we use the parameter values $g = 9.8 \text{ ms}^{-2}$, $\sigma = 1.2 \text{ s}^{-1}$ and $\zeta_0 = 1 \text{ m}$.



Fig. 3. Surface displacement along θ = 0 for *a* = 10 m, *d*/*a* = 0.6, *h*/*a* = 0.4.

4.1. Convergence of the series solution

The solution converges quickly, with in general no more than 20 terms needing to be taken in the expansion of ϕ for a convergent solution in both the interior and exterior regions. Fig. 3 shows a surface wave elevation profile for the diffraction problem for a range of values of *m*, demonstrating that the solution converges quickly with *m*. Fig. 4 shows the normed distance between consecutive terms, $\|\zeta_M - \zeta_{M-1}\|$, with the norm defined as

$$\|\zeta_n\| = \sum_{m=0}^M \epsilon_m \mathbf{i}^m \sum_{r=0}^R \chi_m(r) \cos m\theta$$

with *R* being a number (the radial distance from the center of the cylinder), controlling the total area over which the norm is applied. In all the test results presented in this paper, it is chosen to be six times the cylinder radius *a*. This normed distance gives a good measure of the convergence of the series solution.

Fig. 4 shows a rapid decay of the normed distance between consecutive terms as the number of terms included in the calculation increases. This demonstrates a fast convergence rate using our approach. As the figure shows, taking M = 25



Fig. 4. Error $\|\zeta_m - \zeta_{m-1}\|$ as a function of *m*, for *a* = 10 m.



Fig. 5. Amplitude of interior cylinder oscillation for m = 0, d/a = 0.6. The solid line shows our solution, the circles show Garrett's solution.



Fig. 6. Amplitude of interior cylinder oscillation for m = 1, d/a = 0.6.



Fig. 7. Amplitude of interior cylinder oscillation for m = 2, d/a = 0.6.



Fig. 8. Amplitude of interior cylinder oscillation for m = 3, d/a = 0.6.

terms is sufficient to guarantee a difference of no more than 10^{-5} m between successive terms. By our choice of $\zeta_0 = 1$ m, this implies a relative error of the same order of magnitude, which should be acceptable for any practical application.

4.2. Comparison with previous results

We compare our results with those obtained by Garrett, who defines

$$A_m = \mathscr{F}_{mk} \frac{Z'_k(d)}{k J'_m(ka)},\tag{28}$$

which is the coefficient of $J'_m(kr)$ in the expansion of $\chi_m(r)$ in $r \leq a$. Like Garrett, we plot A_m against the dimensionless cylinder radius ka for constant ocean depth d/a = 0.6 and two different cylinder submersions h/a = 0.4, 0.2. We show the results for m = 0, 1, 2, 3 in Figs. 5–8. In each figure, the solid line represents our direct solution, and the circles show Garrett's solution.

The solution for h/a = 0.2 is excellent with both peaks matching those on Garrett's graphs well, while the second peak for the h/a = 0.4 solution matches less accurately with Garrett's solution. For intermediate values of ka between the two peaks



Fig. 9. $\frac{\partial \phi}{\partial r}$ as a function of z for different values of m. The solution converges quickly in both cases.

(roughly 2 < ka < 4) our solution curve is slightly lower than Garrett's. We believe this is because of the boundary condition matching, which we find to be dependent upon cylinder submergence.

4.3. Matching the boundary conditions

The sharp boundary at the bottom of the cylinder, at z = h and r = a, creates a discontinuity in (16) at this point. As we are using a Fourier series expansion we expect to observe Gibbs phenomenon [11] occurring at the discontinuity. Measuring the severity of the Gibbs phenomenon at the singularity at r = a and z = h will provide us with information about how accurate our solution is.

Fig. 9(a) and (b) shows plots of $\frac{\partial b}{\partial r}$ at r = a and $0 \le z \le d$ for a = 10 m and a = 60 m, respectively. In Fig. 9(a) the solution curves are visually indistinguishable from each other after six terms, and in Fig. 9(b) are visually indistinguishable after 16 terms. The figures show that although the solution converges quickly in both cases, the boundary condition (4) is matched imperfectly. We find the quality of the matching to the boundary condition to be dependent upon cylinder submergence, as demonstrated in Fig. 10(a) and (b). The figures show the error per unit submergence depth

$$\frac{\sum_{z \in [h,d]} \left| \frac{\partial \psi(a,z)}{\partial r} \right|}{d-h}$$

as a function of cylinder height off the ocean floor h. In general the deeper the submergence, the better the matching to the boundary condition at r = a. In both cases, there is a sharp exponential increase in accuracy for submergences greater than 90% of the ocean depth, that is for cylinder heights lower than h/a = 0.1.

When the approach presented here is applied to study other cases, e.g., cases with short-crested incident waves, or cases with the chamber of OWC being of a shape other than a perfect circular shape, or even cases with an uneven bottom bathometry, the total number of terms needed to generate a convergent solution may vary too. The satisfaction of the boundary conditions on the vertical wall of an OWC device should always be examined as a measure for the accuracy of the solution.

Finally, we present example surface wave plots in Figs. 11 and 12. The figures show the diffracted wave heights only for a = 60 m and h/a = 0.4. In both cases plane waves are incident from $-\infty$ along the *x*-axis.

5. Conclusions

A first-order analytical solution for the diffraction of ocean surface waves around a hollow suspended cylinder in an ocean of finite constant depth has been derived and presented in closed form. We have formulated the problem using the theory presented by Garrett [2], however, have used a different and more direct approach to calculate the numerical solution. By writing the problem as a system containing an infinite number of linear equations which we truncate after a suitable number of terms, we obtain an approximation to the solution that agrees well with Garrett's. We find our solution more direct and computationally easier to implement than Garrett's, with good results still provided using this technique.



Fig. 10. Boundary condition matching error over submersion depth as a function of h/a.



Fig. 11. Contour plot showing diffracted wave elevation for *a* = 60 m.



Fig. 12. Contour plot showing wave elevation inside cylinder for *a* = 60 m.

Numerical results are presented for our solution as well as an analysis of the rate of convergence and accuracy of the solution in terms how well it matches the boundary conditions. We find that the rate of convergence is very rapid, with less than 30 terms needing to be taken in the series solution to ensure convergence. The boundary conditions are also met well, with the only numerical issues arising from a singularity in the solution at the bottom of the solid wall of the cylinder. This introduces Gibbs phenomenon at the bottom of the cylinder, the effect of which we find to be dependent upon the cylinder submersion. In general, we find a dramatic improvement in the boundary condition matching error for deeper cylinder submersion, with the error decreasing greatly for submersions deeper than 5/ 6ths of the total ocean depth.

Having obtained a solution for the diffraction problem, a logical extension of the theory is to the coupled diffraction-radiation problem. This will require the formulation and solution for radiating waves produced by an oscillating surface pressure, which may be non-uniformly distributed inside a hollow suspended cylinder in an ocean of finite constant depth. By obtaining a solution for this problem in the same closed form as the one we have obtained for the diffraction problem, we will be able to express the solution to the combined diffraction-radiation problem as the simple linear superposition of the two types of waves. Another possible extension of the research is of course to try to work out the diffraction-radiation problem in the nonlinear wave region or to take into consideration of the variation of bottom topography. Either of these new directions require considerable effort and will be left for the future research.

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