Math 20 Practice exam questions

(1) ~~Evaluate~~

(a)

(i) $\int \frac{(3\ln x + 2)^4}{x}\,dx$ let $u = 3\ln x + 2$

$du = \frac{3}{x}\,dx.$

$= \frac{1}{3}\int (3\ln x + 2)^4 \left(\frac{3}{x}\,dx\right)$

$= \frac{1}{3}\int u^4\,du$

$= \frac{1}{3}\frac{u^5}{5} + c$

$= \frac{1}{15}(3\ln x + 2)^5 + c.$

(ii) $\int e^{3x^2+4}\,x\,dx$ let $u = 3x^2 + 4$

$du = 6x\,dx.$

$= \frac{1}{6}\int e^{3x^2+4}(6x)\,dx$

$= \frac{1}{6}\int e^u\,du$

$= \frac{1}{6}e^u + c$

$= \frac{1}{6}e^{3x^2+4} + c.$

(iii) $\int \overset{u}{x^2}\,\overset{v'}{e^{x/2}}\,dx$ $u = x^2$ $v = 2e^{x/2}$

$u' = 2x$ $v' = e^{x/2}$

$= uv - \int v u' \, dx$

$= 2x^2 e^{x/2} - 2\int \overset{u}{x}\,\overset{v'}{e^{x/2}}\,dx$ $u = x$ $v = 2e^{x/2}$.

$u' = 1$ $v' = e^{x/2}$

$= 2x^2 e^{x/2} - 2\left[uv - \int vu' \, dx\right]$

$= 2x^2 e^{x/2} - 4xe^{x/2} + 4\int e^{x/2} \, dx$

$= 2x^2 e^{x/2} - 4xe^{x/2} + 8e^{x/2} + c.$

(b)

(i) $\iint_R xy \, dy \, dx$ $0 \le x \le 1, \quad 0 \le y \le 2x$

$= \int_0^1 \int_0^{2x} xy \, dy \, dx$

$= \int_0^1 \left.\frac{xy^2}{2}\right|_0^{2x} dx$

$= \int_0^1 \frac{4x \cdot x^2}{2} - 0 \, dx$

$= \int_0^1 2x^3 \, dx$

$= \left.\frac{2x^4}{4}\right|_0^1 = \frac{1}{2} x^4 \Big|_0^1 = \frac{1}{2}.$

(ii) $\iint\limits_R y e^{y^2+x}\, dy\, dx \qquad 0 \le x \le 1, \quad 0 \le y \le 1$

$= \int_0^1 \int_0^1 y e^{y^2+x}\, dy\, dx$

$= \int_0^1 \int_0^1 y e^{y^2+x}\, dx\, dy$

$= \int_0^1 y e^{y^2+x} \Big|_0^1 dy$

$= \int_0^1 \left(y e^{y^2+1} - y e^{y^2} \right) dy \qquad$ let $u_1 = y^2+1 \quad u_2 = y^2$

$du_1 = du_2 = 2y\, dy.$

$= 2\int_{y=0}^{y=1} e^{u_1}\, du_1 - 2\int_{y=0}^{y=1} e^{u_2}\, du_2$

$= 2e^{y^2+1}\Big|_0^1 - 2e^{y^2}\Big|_0^1$

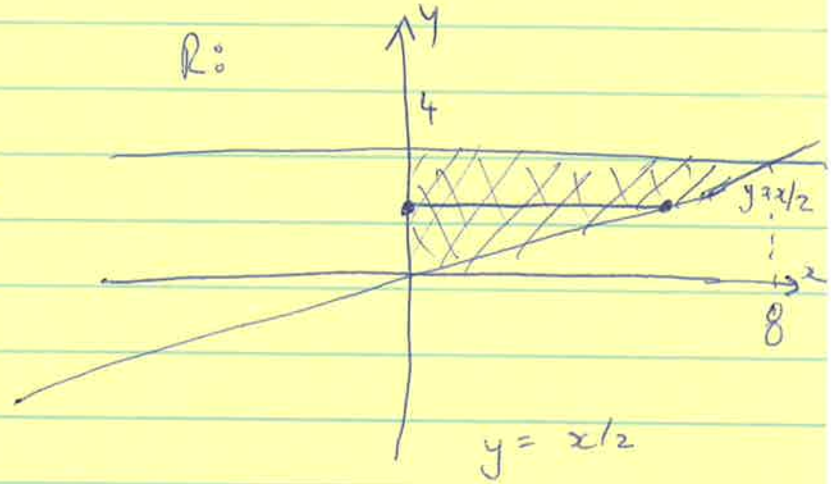
$= 2e^2 - 2e - 2e + 2.$

$= 2e^2 - 4e + 2.$

(iii)

$$\int_0^8 \int_{x/2}^4 \sqrt{y^2+4}\, dy\, dx$$

R:



Changing limits, this becomes...

$y = x/2$

$\Rightarrow x = 2y.$

$$\int_0^4 \int_0^{2y} \sqrt{y^2+4}\, dx\, dy.$$

$$= \int_0^4 x\sqrt{y^2+4}\Big|_{x=0}^{x=2y} dy.$$

$$= \int_0^4 2y\sqrt{y^2+4} - 0\, dy.$$

let $u = y^2+4$

$du = 2y$

$y=0, \; u=4$

$y=4 \quad u=20$

$$= \int_4^{20} \sqrt{u}\, du.$$

$$= \frac{u^{3/2}}{3/2}\Big|_4^{20}$$

$$= \frac{2}{3}\left(20^{3/2} - 8\right).$$

(c) $\displaystyle\int_1^b \frac{16}{x^2}\,dx = \int_1^b 16x^{-2}\,dx$

$$= \left.\frac{16x^{-1}}{-1}\right|_1^b$$

$$= \frac{-16}{b} + 16$$

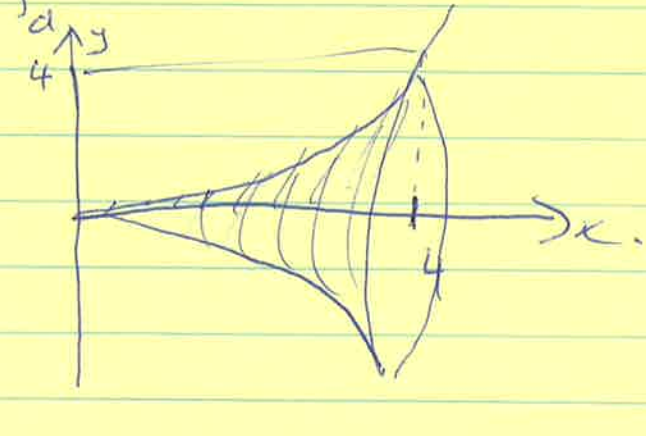
So we need $-\dfrac{16}{b} + 16 = 14.$

$$\frac{16}{b} = 2.$$

$$b = \frac{16}{2} = 8$$

Note: This is the same procedure as we would use to find the median m of a probability distribution, which is the value for which

$$\int_a^m f(x)\,dx = 0.5.$$



(d)(i) $f(x) = x^2/4.$

$$V = \int_0^4 2\pi f(x)^2\,dx$$

$$= 2\pi\int_0^4 \frac{x^4}{16}\,dx$$

$$= \frac{\pi}{8}\int_0^4 x^4\,dx$$

$$= \frac{\pi}{8} \left. \frac{x^5}{5} \right|_0^4$$

$$= \frac{\pi}{40} (4^5).$$

$$\text{(ii) } V = \int_0^1 \int_0^1 x\sqrt{x^2+y}\, dx\, dy$$

let $u = \cancel{x}^2 + y$

$du = 2x\, dx.$

$x = 0,\ u = y$

$x = 1 \quad u = y+1$

$$= \int_0^1 \frac{1}{2} \int_y^{y+1} 2x\sqrt{x^2+y}\, dx\, dy.$$

$$= \int_0^1 \frac{1}{2} \int_y^{y+1} \sqrt{u}\, du.\ dy.$$

$$= \int_0^1 \frac{1}{2} \left. \frac{u^{3/2}}{3/2} \right|_y^{y+1} dy.$$

$$= \int_0^1 \frac{1}{3} \left((y+1)^{3/2} - y^{3/2} \right) dy.$$

$$= \frac{1}{3} \left[\frac{(y+1)^{5/2}}{5/2} - \frac{y^{5/2}}{5/2} \right]_0^1$$

$$= \frac{2}{15} \left[(2^{5/2} - 1) - (1 - 0) \right]$$

$$= \frac{2}{15} (2^{5/2} - 2).$$

2

(a)(i) $f(x,y) = 2x^2 + 4xy + 4y^2 - 3x + 5y - 15$.

$f_x = 4x + 4y - 3 = 0$

$f_y = 4x + 8y + 5 = 0$ for crit pts.

$- \quad -4y - 8 = 0$.

$y = -2$

so $4x - 8 - 3 = 0$.

$x = 11/3$.

$(x,y) = \left(\frac{11}{3}, -2\right)$.

$f_{xx} = 4 > 0$.

$f_{yy} = 8$

$f_{xy} = 4$. so $D = f_{xx} f_{yy} - f_{xy}^2$

$= 32 - 16$

$= 16 > 0$.

so $\left(\frac{11}{3}, -2\right)$ is a minimum.

(ii) $f(x,y) = 7x^2 + y^2 - 3x + 6y - 5xy$.

$f_x = 14x - 3 - 5y = 0$ ...(1)

$f_y = -5x + 6 + 2y = 0$ for crit pt. ...(2)

(1): $14x = 3 + 5y$

$$x = \frac{3+5y}{14}.$$

Sub into (2): $2y + 6 - 5\left(\frac{3+5y}{14}\right) = 0.$

$$2y + 6 - \frac{15}{14} - \frac{25y}{14} = 0.$$

$$\frac{3}{14}y + 5 - \frac{1}{14} = 0.$$

$$3y + 70 - 1 = 0.$$

$$3y = -69$$

$$y = -23.$$

$$x = \frac{3+5(23)}{14}$$

$$= \frac{118}{14}$$

$f_{xx} = 14 > 0.$

$f_{yy} = 2$

$f_{xy} = -5.$

so $D = (14)(2) - 25.$

$= 3 > 0$ so minimum.

~~(iii) $f(x,y) = 2x^2 + 4xy +$~~

(iii) $f(x,y) = y^2 - 2xy + 4x^3 + 20x^2$

$f_x = -2y + 12x^2 + 40x = 0$ for crit pt

$\Rightarrow y = 6x^2 + 20x$

$f_y = 2y - 2x = 0$ for ~~max.~~ crit pt.

so $y = x$.

$x = 6x^2 + 20x$

$6x^2 + 19x = 0$

$x(6x + 19) = 0$.

$x = 0, \quad x = -\frac{19}{6}$.

so crit pts at $(0,0), \left(-\frac{19}{6}, -\frac{19}{6}\right)$.

$f_{xx} = 24x + 40$

$f_{yy} = 2$

$f_{xy} = -2$.

@ $(0,0)$: $D = (40)(20) - (-2)^2 > 0$ and $f_{xx} > 0$ so min.

@ $\left(-\frac{19}{6}, -\frac{19}{6}\right)$: $D = (-19 \times 4 + 40)(2) - 4 < 0$ and $f_{xx} > 0$ so max

(b)

(i) $\frac{dy}{dx} = 4e^{2x}$. elementary

$dy = 4e^{2x}\, dx$

$\int dy = 4\int e^{2x}\, dx$

$y = 2e^{2x} + c$.

(ii) $\frac{dy}{dx} = x^3 + \frac{7}{x}$

$dy = \left(x^3 + \frac{7}{x}\right) dx$ elementary

$y = \int x^3 + \frac{7}{x}\, dx$

$= \frac{x^4}{4} + 7\ln x + c$.

(iii) $\frac{dy}{dx} = \frac{(e^x + x)}{(y+1)}$

$(y+1)\, dy = (e^x + x)\, dx$ separable

$\int y + 1\, dy = \int e^x + x\, dx$

$\frac{y^2}{2} + y = e^x + \frac{x^2}{2} + c$.

(iv) $x \ln x \frac{dy}{dx} + y = 2x^2.$

$$\frac{dy}{dx} + y\left(\frac{1}{x \ln x}\right) = \frac{2x}{\ln x}$$

linear first order.

$$I(x) = e^{\int p(x)dx}$$

$$= e^{\int \frac{1}{x \ln x} dx}$$

$u = \ln x$
$du = \frac{1}{x} dx$

$$= e^{\int \frac{1}{u} du}$$

$$= e^{\ln u}$$

$$= u$$

$$= \ln(x).$$

so $\frac{d}{dx}\left(y \times \frac{1}{x}\right) = 2x.$

$$\frac{y}{x} = \int 2x\,dx$$

$$= x^2 + c.$$

$$y = x^3 + Cx.$$

(c) $f(x,y) = 48xy - x^2 - 3y^2$.

~~F(x,y)~~ constraint: $x + y = 52$.

$F(x,y,\lambda) = 48xy - x^2 - 3y^2 - \lambda(x+y-52)$.

(1) $F_x = 48y - 2x - \lambda = 0$ for crit. pt.

(2) $F_y = 48x - 6y - \lambda = 0$

(3) $F_\lambda = -(x+y-52) = 0$

From (1): $\lambda = 48y - 2x$

(2): $\lambda = 48x - 6y$.

so

$48y - 2x = 48x - 6y$.

$50x = 54y$.

$x = \frac{54}{50}y$

$= \frac{28}{25}y$.

from (3): $\frac{28}{25}y + y = 52$.

$53y = 1300$

$$\begin{cases} y = \frac{1300}{53} \\ x = \frac{1300}{53} \times \frac{28}{25}. \end{cases}$$

(3) Continuous probability distributions

(a) $f(x) = k\sqrt{x}$

(i) We need $k\int_0^4 \sqrt{x}\,dx = 1.$

$$k \left. \frac{x^{3/2}}{3/2} \right|_0^4 = 1.$$

$$k\left(\frac{8}{\frac{3}{2}} - 0\right) = 1.$$

$$\frac{16k}{3} = 1$$

$$k = \frac{3}{16}.$$

(ii) Need $k\int_4^9 \sqrt{x}\,dx = 1.$

$$k \left. \frac{2x^{3/2}}{3} \right|_4^9 = 1.$$

$$\frac{2k}{3}(27 - 8) = 1.$$

$$\frac{38k}{3} = 1$$

$$k = \frac{3}{38}.$$

(b)

(i) $f(x) = \frac{2}{9}(x-2)$ on $[2,5]$.

$$P(2 < X < 4) = \frac{2}{9}\int_2^4 x-2\,dx$$

$$= \frac{2}{9}\left(\frac{x^2}{2} - 2x\right)\Big|_2^4$$

$$= \frac{2}{9}\left[(8-8)-(2-4)\right]$$

$$= \frac{4}{9}.$$

$$\mathbb{E}(X) = \int_2^5 \frac{2}{9}(x-2)\,x\,dx$$

$= \cdots$

$= 4.$

$$\sigma^2 = \int_2^5 \frac{2}{9}(x-2)\,(x-4)^2\,dx$$

$= \cdots$

$= \frac{1}{2}.$

$$C(x) = \int_2^x f(t)\,dt$$

$$= \int_2^x \frac{2}{9}(t-2)\,dt$$

$$= \frac{1}{9}(x-2)^2.$$

(ii) $f(x) = 5x^{-6}$. on $[1,\infty)$.

$$P(2<x<4) = \int_2^4 5x^{-6}\,dx$$

$$= \cancel{5}\frac{x^{-5}}{-\cancel{5}}\Big|_2^4$$

$$= \left(\frac{1}{4^5} - \frac{1}{2^5}\right)$$

$$\mathbb{E}(X) = \int_1^\infty 5x^{-5}\,dx$$

$$= \frac{-5}{4}x^{-4}\Big|_1^\infty$$

$$= \frac{5}{4}$$

$$\sigma^2 = \int_1^\infty 5x^{-5}\cdot x^2\,dx - (\mathbb{E}(X))^2$$

$$= \int_1^\infty 5x^{-3}\,dx - \frac{25}{16}$$

$$= \frac{-5}{2}x^{-2}\Big|_1^\infty - \frac{25}{16} = \frac{5}{2} - \frac{25}{16}.$$

$$C(x) = \int_1^x 5t^{-6}\,dt$$

$$= -t^{-5}\Big|_1^x = 1 - \frac{1}{x^5}$$

(iii) $f(x) = \dfrac{\sqrt{x}+3}{20\sqrt{x}}$ on $[1,9]$.

$$= \frac{1}{20} + \frac{3}{20}x^{-1/2}$$

$$P(2<X<4) = \int_2^4 \frac{1}{20} + \frac{3}{20}x^{-1/2}\,dx$$

$$= \left[\frac{x}{20} + \frac{3}{10}x^{1/2}\right]_2^4$$

$$= \left(\frac{1}{5} + \frac{6}{10}\right) - \left(\frac{1}{10} + \frac{3\sqrt{2}}{10}\right).$$

$$\mathbb{E}(X) = \int_1^9 \frac{x}{20} + \frac{3}{20}x^{1/2}\,dx$$

$$= \frac{x^2}{40} + \frac{\cancel{3}}{20}\frac{x^{3/2}}{\cancel{3}/2}\Bigg|_1^9$$

$$= \left(\frac{81}{40} + \frac{27}{10}\right) - \left(\frac{1}{40} + \frac{1}{10}\right). \quad = \mu.$$

$$\sigma^2 = \int_1^9 \frac{x^2}{20} + \frac{3x^{3/2}}{20}\,dx \quad - \mu^2.$$

$$= \cdots =$$

$$C(x) = \int_1^x \frac{1}{20} + \frac{3}{20}t^{-1/2}\,dt$$

$$= \left[\frac{t}{20} + \frac{3}{10}\sqrt{t}\right]_1^x = \frac{x}{20} + \frac{3}{10}\sqrt{x} - \frac{1}{20} - \frac{3}{10}.$$

(b) $\int_a^m f(x)\,dx = \frac{1}{2}$.

(i) Need $\frac{2}{9}\int_2^m x-2\,dx = \frac{1}{2}$

$$\left.\frac{x^2}{2} - 2x\right|_2^m = \frac{9}{4}$$

$$\left(\frac{m^2}{2} - 2m\right) + 2 = \frac{9}{4}.$$

ie solve $m^2 - 4m + 4 - \frac{9}{2} = 0.$

$$m^2 - 4m - 0.5 = 0$$

~~$(m-5)(m+1)=0$~~

~~so $m=5$~~

~~$m=-1$ is outside of $[2,5]$, so the median is $m=5$.~~

$$m = \frac{4 \pm \sqrt{16+2}}{2}$$

$$= 2 \pm \frac{\sqrt{18}}{2}.$$

and $m = 2 - \frac{\sqrt{18}}{2}$ is outside of $[2,5]$, so median is $m = 2 + \frac{\sqrt{18}}{2}$.

(ii) need $\int_1^m 5x^{-6}\,dx = \frac{1}{2}$

$$\left.-x^{-5}\right|_1^m = \frac{1}{2}$$

$$1 - \frac{1}{m^5} = \frac{1}{2}$$

$$\frac{1}{m^5} = \frac{1}{2}$$

$m^5 = 2$

so $m = 2^{1/5}$..

(iii) Need $\displaystyle\int_1^m \frac{1}{20} + \frac{3}{20} x^{-1/2} \, dx = \frac{1}{2}.$

$$\left(\frac{x}{20} + \frac{3}{10} x^{1/2} \right) \Bigg|_1^m = \frac{1}{2}.$$

ie solve: $\frac{m}{20} + \frac{3}{10}\sqrt{m} - \frac{1}{2} - \frac{3}{10} = \frac{1}{2}$. [This is a bit messy....]

These are all different to the respective means of each distribution, because each distribution is <u>non-symmetric</u>!

(c). See the textbook, Sec. 11.3!