Midterm II

(1)

(a) $\int_0^\infty \frac{x}{(x^2+2)^2}\,dx = \lim_{a\to\infty} \int_0^a \frac{x}{(x^2+2)^2}\,dx$

$u = x^2+2$
$du = 2x\,dx$
$x\,dx = \frac{1}{2}\,du$

$x = a \quad u = a^2+2$
$x = 0 \quad u = 2.$

$$= \lim_{a\to\infty} \frac{1}{2} \int_2^{a^2+2} \frac{du}{u^2}$$

$$= \lim_{a\to\infty} \frac{1}{2} \frac{u^{-1}}{-1}\Big|_2^{a^2+2}$$

$$= -\frac{1}{2} \lim_{a\to\infty} \left(\cancel{\frac{1}{a^2+2}} - \frac{1}{2}\right).$$

$$= -\frac{1}{2}\left(-\frac{1}{2}\right)$$

$$= \frac{1}{4}.$$

(b) $\iint_R (3x^2+4y)\,dx\,dy$

$0 \le x \le 3$
$1 \le y \le 4$

$$= \int_1^4 \int_0^3 3x^2+4y\,dx\,dy$$

$$= \int_1^4 x^3+4xy\Big|_0^3\,dy$$

$$= \int_1^4 27+12y\,dy$$

$$= 27y + \frac{12y^2}{2} \Big|_1^4$$

$$= 27y + 6y^2 \Big|_1^4$$

$$= (108 + 96) - (27 + 6)$$

$$= 204 - 33 = 171$$

$$\cancel{= 181}$$

2

(a) $f(x,y) = (y-3)^2 e^{x+2y}$

$$f_x(x,y) = (y-3)^2 e^{x+2y} = f(x,y).$$

$$f_y(x,y) = uv' + vu' \qquad u = (y-3)^2 \quad v = e^{x+2y}$$

$$u' = 2(y-3) \quad v' = 2e^{x+2y}$$

$$= 2(y-3)^2 e^{x+2y} + 2(y-3)e^{x+2y}$$

$$= 2(y-3)e^{x+2y}(y-3+1)$$

$$= 2(y-3)(y-2)e^{x+2y}.$$

(e) $f(x,y) = \ln|5x-7y|$

$$f_x = \frac{5}{5x-7y} \qquad f_y = \frac{-7}{5x-7y}$$

$$\cancel{f_{xx} = -25}$$

$$= 5(5x-7y)^{-1} \qquad = -7(5x-7y)^{-1}$$

$$\therefore f_{xx} = -5(5x-7y)^{-2}(5) \qquad f_{yy} = 7(5x-7y)^{-2}(-7)$$

$$= \frac{-25}{(5x-7y)^2} \qquad = \frac{-49}{(5x-7y)^2}$$

$$f_{xy} = -5(5x-7y)^{-2}(-7)$$

$$= \frac{35}{(5x-7y)^2} = f_{yx}\,!$$

(f) $B = \dfrac{703w}{h^2}$

(i) $B_w = \dfrac{\partial B}{\partial w} = \dfrac{703}{h^2}$ means the rate at which your BMI changes as your weight changes (Note: if w increases so does BMI) as $\dfrac{703}{h^2} > 0$).

$$B_h = \frac{\partial B}{\partial h} = \frac{-703w \times 2}{h^3}$$

$$= \frac{-1406w}{h^3} < 0$$ means the rate at which BMI changes as h increases (BMI goes down as h goes up, because $B_h < 0$).

(ii) $w = 317$, $h = 6 \times 12 + 7 = 79$

so $$BMI = \frac{703(317)}{(79)^2}$$

$= 35.71$, so Jake Long is OBESE !

(e) $f(x,y) = x^2(y+1)^2 + k(x+1)^2y^2$, $(x,y) = (0,0)$.

(i) $(0,0)$ is a critical pt when $f_x(0,0) = f_y(0,0) = 0$.

$f_x = 2x(y+1)^2 + 2k(x+1)y^2$

$f_x(0,0) = 0(0+1)^2 + 2k(0+1)0^2 = 0.$

$f_y = 2x^2(y+1) + 2k(x+1)^2 y.$

$f_y(0,0) = 0 + 2k(1)(0) = 0$, so $(0,0)$ is always a critical point, for all values of k!

1 (ii) For a relative min we need $f_{xx}(0,0) > 0$ and $D(0,0) > 0$.

1 $f_{xx} = 2(y+1)^2 + 2ky^2$

$f_{xx}(0,0) = 2 + 2k(0) = 2 > 0$ always.

1 $f_{yy} = 2x^2 + 2k(x+1)^2$

$f_{yy}(0,0) = 0 + 2k(0+1)^2 = 2k.$

1 $f_{xy} = 4x(y+1) + 4k(x+1)y.$

$f_{xy}(0,0) = 4(0)(0+1) + 4k(0+1)(0) = 0.$

1 So $D(0,0) = f_{xx}(0,0)\,f_{yy}(0,0) - \left(f_{xy}(0,0)\right)^2$

$= 2 \times 2k - 0^2$

$= 4k.$

$D(0,0) > 0$ when $k > 0$, so $(0,0)$ is a relative min when $\boxed{k > 0.}$

(c) $f(x,y) = 12x^{3/4}y^{1/4}$

constraint is $180x + 100y = 25200$.

or $g(x,y) = 180x + 100y - 25200 = 0$.

Lagrange function:

1 $F(x,y,\lambda) = 12x^{3/4}y^{1/4} - \lambda(180x + 100y - 25200)$.

1 let $F_x = \frac{3}{4}(12)x^{-1/4}y^{1/4} - 180\lambda = 0$ ... (1)

1 $F_y = \frac{12}{4}x^{3/4}y^{-3/4} - 100\lambda = 0$ ... (2)

1 $F_\lambda = -180x - 100y + 25200 = 0$ ... (3)

(1): $\cancel{9}x^{-1/4}y^{1/4} = \overset{20}{\cancel{180}}\lambda$

$$\lambda = \frac{x^{-1/4}y^{1/4}}{20}$$

(2): $3x^{3/4}y^{-3/4} = 100\lambda$

$$\lambda = \frac{3x^{3/4}y^{-3/4}}{100}.$$

1 equating (1) & (2):

$$\frac{x^{-1/4}y^{1/4}}{20} = \frac{3x^{3/4}y^{3/4}}{100}$$

$$x^{-1/4} y^{1/4} = \frac{3x^{3/4} y^{-3/4}}{5}.$$

$$y^{1/4} = \frac{3x\, y^{-3/4}}{5}.$$

$$y = \frac{3x}{5}.$$

sub into (3):

$$-18\cancel{0}x - 10\cancel{0}\left(\frac{3x}{5}\right) + 2520\cancel{0} = 0.$$

$$18x + \frac{\overset{6}{3\cancel{0}}x}{\cancel{5}} = 2520.$$

$$24x = 2520.$$

$$x = 105.$$

so $$y = \frac{3(105)}{5}$$

$$= 63.$$

So output is maximised when $x = 105$ units of labor & $y = 63$ units of capital.